

BARGMANN-TYPE TRANSFORMS AND MODIFIED HARMONIC OSCILLATORS

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ABSTRACT. We study some complete orthonormal systems on the real-line. These systems are determined by Bargmann-type transforms, which are Fourier integral operators with complex-valued quadratic phase functions. Each system consists of eigenfunctions for a second-order elliptic differential operator like the Hamiltonian of the harmonic oscillator. We also study the commutative case of a certain class of systems of second-order differential operators called the non-commutative harmonic oscillators. By using the diagonalization technique, we compute the eigenvalues and eigenfunctions for the commutative case of the non-commutative harmonic oscillators. Finally, we study a family of functions associated with an ellipse in the phase plane. We show that the family is a complete orthogonal system on the real-line.

1. INTRODUCTION

We are concerned with the Bargmann-type transform defined by

$$T_h u(z) = C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} u(x) dx, \quad z \in \mathbb{C}, \quad (1)$$

where $\phi(z, x)$ is a complex-valued quadratic phase function of the form

$$\phi(z, x) = \frac{A}{2} z^2 + Bzx + \frac{C}{2} x^2, \quad A, B, C \in \mathbb{C}$$

with assumptions $B \neq 0$ and $\text{Im } C > 0$, and $C_\phi = 2^{-1/2} \pi^{-3/4} |B| (\text{Im } C)^{-1/4}$. Throughout of the present paper, we deal with only the one-dimensional case for the sake of simplicity. It is possible to discuss higher dimensional case, and we omit the detail. Note that the integral transform (1) is well-defined for tempered distributions on \mathbb{R} since $\text{Re}(i\phi(z, x)) = \mathcal{O}(-\text{Im } Cx^2/3)$ for $|x| \rightarrow \infty$.

These integral transforms were introduced by Sjöstrand (See, e.g., [9]). He developed microlocal analysis based on them. One can see (1) as a global Fourier integral operator associated with a linear canonical transform $\kappa_T : \mathbb{C}^2 \ni (x, -\phi'_x(z, x)) \mapsto (z, \phi'_z(z, x)) \in \mathbb{C}^2$, that is,

$$\kappa_T : \mathbb{C}^2 \ni (x, \xi) \mapsto \left(-\frac{Cx + \xi}{B}, Bx - \frac{A(Cx + \xi)}{B} \right) \in \mathbb{C}^2. \quad (2)$$

If we set $\Phi(z) = \max_{x \in \mathbb{R}} \text{Re}(i\phi(z, x))$, then we have

$$\begin{aligned} \Phi(z) &= \frac{|Bz|^2}{4 \text{Im } C} - \text{Re} \left\{ \frac{(Bz)^2}{4 \text{Im } C} + \frac{Az^2}{2i} \right\}, \\ \kappa_T(\mathbb{R}^2) &= \left\{ \left(z, \frac{2}{i} \frac{\partial \Phi}{\partial z}(z) \right) \mid z \in \mathbb{C} \right\}. \end{aligned} \quad (3)$$

This means that the singularities of a tempered distribution u described in the phase plane \mathbb{R}^2 are translated into those of $T_h u$ in the I-Lagrangian submanifold $\kappa_T(\mathbb{R}^2)$. The microlocal analysis of

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Sjöstrand is based on the equivalence of the Weyl quantization on \mathbb{R} , the Weyl quantization on $\kappa_T(\mathbb{R})$, and the Berezin-Toeplitz quantization on \mathbb{C} . For more detail about them, see [9] or [2].

Let $L^2(\mathbb{R})$ be the set of all square-integrable functions on \mathbb{R} , and let $L^2_\Phi(\mathbb{C})$ be the set of all square-integrable functions on \mathbb{C} with respect to a weighted measure $e^{-2\Phi(z)/h}L(dz)$, where L is the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. We denote by $\mathcal{H}_\Phi(\mathbb{C})$ the set of all entire functions in $L^2_\Phi(\mathbb{C})$. It is well-known that T_h gives a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $\mathcal{H}_\Phi(\mathbb{C})$, that is,

$$(T_h u, T_h v)_{L^2_\Phi} = (u, v)_{L^2} \quad u, v \in L^2(\mathbb{R}),$$

where

$$(U, V)_{L^2_\Phi} = \int_{\mathbb{C}} U(z) \overline{V(z)} e^{-2\Phi(z)/h} L(dz), \quad U, V \in L^2_\Phi(\mathbb{C}),$$

$$(u, v)_{L^2} = \int_{\mathbb{R}} u(x) \overline{v(x)} dx \quad u, v \in L^2(\mathbb{R}).$$

We sometimes denote $(U, V)_{L^2_\Phi}$ for $U, V \in \mathcal{H}_\Phi(\mathbb{C})$ by $(U, V)_{\mathcal{H}_\Phi}$. The inverse mapping T_h^* is given by

$$T_h^* U(x) = C_\Phi \int_{\mathbb{C}} e^{-i\overline{\phi(z, x)}/h} U(z) e^{-2\Phi(z)/h} L(dz), \quad x \in \mathbb{R}, \quad U \in \mathcal{H}_\Phi(\mathbb{C}).$$

Note that T_h^* is well-defined for $U \in L^2_\Phi(\mathbb{C})$. $T_h \circ T_h^*$ becomes an orthogonal projector of $L^2_\Phi(\mathbb{C})$ onto $\mathcal{H}_\Phi(\mathbb{C})$. More concretely,

$$T_h \circ T_h^* U(z) = \frac{C_\Phi}{h} \int_{\mathbb{C}} e^{2\Psi(z, \zeta)/h} U(\zeta) e^{-2\Phi(\zeta)/h} L(d\zeta), \quad U \in L^2_\Phi(\mathbb{C}), \quad (4)$$

where $C_\Phi = |B|^2/(2\pi \operatorname{Im} C)$, and $\Psi(z, \zeta)$ is a holomorphic quadratic function defined by the critical value of $-\{\phi(z, X) - \phi(\zeta, \bar{X})\}/2i$ for $X \in \mathbb{C}$, that is,

$$\Psi(z, \zeta) = \frac{|B|^2 z \zeta}{4 \operatorname{Im} C} - \frac{B^2 z^2 + \bar{B}^2 \zeta^2}{8 \operatorname{Im} C} - \frac{A z^2 - \bar{A}^2 \zeta^2}{4i}. \quad (5)$$

In particular, $U = T_h \circ T_h^* U$ for $U \in \mathcal{H}_\Phi(\mathbb{C})$, and $\mathcal{H}_\Phi(\mathbb{C})$ becomes a reproducing kernel Hilbert space.

Here we recall elementary facts related with the classical Bargmann transform B_h which is the most important example of T_h . We can refer [4] for the description below. The Bargmann transform B_h on \mathbb{R} is defined by

$$B_h(z) = 2^{-1/2} (\pi h)^{-3/4} \int_{\mathbb{R}} e^{-(z^2/4 - zx + x^2/2)/h} u(x) dx, \quad z \in \mathbb{C}.$$

We denote $L_\Phi(\mathbb{C})$ and $\mathcal{H}_\Phi(\mathbb{C})$ for B_h by $L_B^2(\mathbb{C})$ and $\mathcal{H}_B(\mathbb{C})$ respectively, those are,

$$L_B^2(\mathbb{C}) = \left\{ U(z) \mid \int_{\mathbb{C}} |U(z)|^2 e^{-|z|^2/2h} L(dz) < \infty \right\},$$

and $\mathcal{H}_B(\mathbb{C}) = \{U(z) \in L_B^2(\mathbb{C}) \mid \partial U / \partial \bar{z} = 0\}$. The Bargmann projector, which is the orthogonal projection of $L_B^2(\mathbb{C})$ onto $\mathcal{H}_B(\mathbb{C})$, is given by

$$B_h \circ B_h^* U(z) = \frac{1}{2\pi h} \int_{\mathbb{C}} e^{z\bar{\zeta}/2h} U(\zeta) e^{-|\zeta|^2/2h} L(d\zeta), \quad U \in L_B^2(\mathbb{C}).$$

In view of the Taylor expansion of the reproducing kernel $e^{z\bar{\zeta}/2h}/(2\pi h)$, the formula $U = B_h \circ B_h^* U$ for $U \in \mathcal{H}_B(\mathbb{C})$ becomes

$$U(z) = \sum_{n=0}^{\infty} (U, \varphi_{B,n})_{\mathcal{H}_B} \varphi_{B,n}(z), \quad \varphi_{B,n}(z) = \frac{z^n}{\sqrt{\pi(2h)^{n+1}n!}}, \quad n = 0, 1, 2, \dots$$

A family of functions $\{\varphi_{B,n}\}_{n=0}^{\infty}$ is a complete orthonormal system of $\mathcal{H}_B(\mathbb{C})$ since $\mathcal{H}_B(\mathbb{C})$ is the set of all entire functions belonging to $L_B^2(\mathbb{C})$.

We shall see more detail about $\{\varphi_{B,n}\}_{n=0}^{\infty}$. We set for $U \in \mathcal{H}_B(\mathbb{C})$

$$\Lambda_B U(z) := 2h \frac{\partial U}{\partial z}(z), \quad \Lambda_B^* U(z) := zU(z).$$

Actually, Λ_B^* is the adjoint of Λ_B on $\mathcal{H}_B(\mathbb{C})$. Elementary computation gives

$$(\Lambda_B^* \circ \Lambda_B + h)\varphi_{B,n} = (\Lambda_B \circ \Lambda_B^* - h)\varphi_{B,n} = (2n+1)h\varphi_{B,n}, \quad n = 0, 1, 2, \dots \quad (6)$$

We shall pull back these facts on \mathbb{R} by using B_h^* . Set

$$\phi_{B,n}(x) := B_h^* \varphi_{B,n}(x), \quad P_B := B_h^* \circ \Lambda_B \circ B_h, \quad P_B^* := B_h^* \circ \Lambda_B^* \circ B_h.$$

$\phi_{B,n}$ is said to be the n -th Hermite function, and a family $\{\phi_{B,n}\}_{n=0}^{\infty}$ is a complete orthonormal system of $L^2(\mathbb{R})$ since B_h is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $\mathcal{H}_B(\mathbb{C})$. Operators

$$P_B = h \frac{d}{dx} + x, \quad P_B^* = -h \frac{d}{dx} + x$$

are said to be annihilation and creation operators respectively. Note that

$$\varphi_{B,n} = ((2h)^n n!)^{-1/2} (\Lambda_B^*)^n \varphi_{B,0}, \quad n = 0, 1, 2, \dots,$$

$$\phi_{B,0}(x) := B_h^* \varphi_{B,0}(x) = \frac{1}{(\pi h)^{1/4}} e^{-x^2/2h}.$$

Then we have so-call the Rodrigues formula

$$\phi_{B,n}(x) = \frac{1}{\sqrt{(2h)^n n!}} (P_B^*)^n \phi_{B,0}(x) = \frac{(-1)^n}{(\pi h)^{1/4} \sqrt{(2h)^n n!}} e^{-x^2/2h} \left(h \frac{d}{dx} \right)^n e^{-x^2/h}.$$

Set $H_B = P_B^* \circ P_B + h = P_B \circ P_B^* - h$. Then

$$H_B = -h^2 \frac{d^2}{dx^2} + x^2$$

which is said to be the Hamiltonian of the harmonic oscillator. The equation (6) becomes

$$H_B \phi_{B,n} = (2n+1)h \phi_{B,n}, \quad n = 0, 1, 2, \dots$$

Thus the n -th Hermite function $\phi_{B,n}$ is an eigenfunction of H_B for the n -th eigenvalue $(2n+1)h$.

The purpose of the present paper is to study the generalization of the known facts on the usual Bargmann transform B_h . The plan of this paper is as follows. In Section 2 we study the general Bargmann-type transform (1), and obtain generalized annihilation and creation operators, the Hamiltonian of the generalized harmonic oscillator and its eigenvalues, generalized Hermite functions and the Rodrigues formula. In Section 3 we study a 2×2 -system of second-order ordinary differential operators, which is said to be a non-commutative harmonic oscillators. More precisely, we study the commutative case of the non-commutative harmonic oscillators, and obtain the eigenvalues and eigenfunctions by using our original elementary computation. Finally in Section 4 we study the general Bargmann-type transform (1) which might be related with ellipses in the phase plane \mathbb{R}^2 .

2. MODIFIED HARMONIC OSCILLATORS AND HERMITE FUNCTIONS

In this section we study the general form of the Bargmann-type transform (1). We remark that the choice of the constant A in the phase function is not essential. We can choose

$$A = -\frac{iB^2}{2\operatorname{Im} C}.$$

Then (3) and (5) become very simple as

$$\Phi(z) = \frac{|Bz|^2}{4\operatorname{Im} C}, \quad \Psi(z, \bar{\zeta}) = \frac{Bz \cdot \overline{B\zeta}}{4\operatorname{Im} C}$$

respectively. Moreover, the orthogonal projector (4) of $L_{\Phi}^2(\mathbb{C})$ onto $\mathcal{H}_{\Phi}(\mathbb{C})$, and the I-Lagrangian submanifold (2) become

$$T_h \circ T_h^* U(z) = \frac{C_{\Phi}}{h} \int_{\mathbb{C}} e^{Bz \cdot \overline{B\zeta}/2h \operatorname{Im} C} U(\zeta) e^{-|Bz|^2/2h \operatorname{Im} C} L(d\zeta),$$

$$\kappa_T(\mathbb{R}^2) = \left\{ \left(z, \frac{|B|^2}{2i \operatorname{Im} C} \bar{z} \right) \mid z \in \mathbb{C} \right\}$$

respectively. Recall $U(z) = T_h \circ T_h^* U(z)$ for all $U \in \mathcal{H}_{\Phi}(\mathbb{C})$. If we consider the Taylor expansion of $e^{-|Bz|^2/2h \operatorname{Im} C}$, we have for $U \in \mathcal{H}_{\Phi}(\mathbb{C})$,

$$U(z) = \sum_{n=0}^{\infty} (U, \varphi_n)_{\mathcal{H}_{\Phi}} \varphi_n(z),$$

$$\varphi_n(z) = \frac{|B|}{\sqrt{2\pi h \operatorname{Im} C}} \cdot \frac{1}{\sqrt{n!}} \cdot \left(\frac{Bz}{\sqrt{2h \operatorname{Im} C}} \right)^n, \quad n = 0, 1, 2, \dots$$

Theorem 2.1. *The family of monomials $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthonormal system of $\mathcal{H}_{\Phi}(\mathbb{C})$.*

Proof. The completeness is obvious. We have only to show that $(\varphi_m, \varphi_n)_{\mathcal{H}_{\Phi}} = \delta_{mn}$, where δ_{mn} is Kronecker's delta. Without loss of generality we may assume that $m \geq n$. By using the integration by parts and the change of variable $\zeta = Bz/\sqrt{h \operatorname{Im} C}$, we deduce that

$$\begin{aligned} (\varphi_m, \varphi_n)_{\mathcal{H}_{\Phi}} &= \frac{|B|^2}{2\pi h \operatorname{Im} C} \cdot \frac{1}{\{m!n!(2h \operatorname{Im} C)^{m+n}\}^{1/2}} \\ &\quad \times \int_{\mathbb{C}} (Bz)^m (\overline{Bz})^n e^{-|Bz|^2/2h \operatorname{Im} C} L(dz) \\ &= \frac{|B|^2}{2\pi h \operatorname{Im} C} \cdot \frac{1}{\{m!n!(2h \operatorname{Im} C)^{m+n}\}^{1/2}} \\ &\quad \times \int_{\mathbb{C}} \left\{ \left(-\frac{2h \operatorname{Im} C}{\overline{B}} \frac{\partial}{\partial \bar{z}} \right)^m e^{-|Bz|^2/2h \operatorname{Im} C} \right\} (\overline{Bz})^n L(dz) \\ &= \frac{|B|^2}{2\pi h \operatorname{Im} C} \cdot \frac{1}{\{m!n!(2h \operatorname{Im} C)^{m+n}\}^{1/2}} \\ &\quad \times \int_{\mathbb{C}} \left\{ \left(-\frac{2h \operatorname{Im} C}{\overline{B}} \frac{\partial}{\partial \bar{z}} \right)^m (\overline{Bz})^n \right\} e^{-|Bz|^2/2h \operatorname{Im} C} L(dz) \\ &= \delta_{mn} \frac{|B|^2}{2\pi h \operatorname{Im} C} \cdot \frac{1}{m!(2h \operatorname{Im} C)^m} \\ &\quad \times \int_{\mathbb{C}} \left\{ \left(-\frac{2h \operatorname{Im} C}{\overline{B}} \frac{\partial}{\partial \bar{z}} \right)^m (\overline{Bz})^m \right\} e^{-|Bz|^2/2h \operatorname{Im} C} L(dz) \\ &= \delta_{mn} \frac{|B|^2}{2\pi h \operatorname{Im} C} \int_{\mathbb{C}} e^{-|Bz|^2/2h \operatorname{Im} C} L(dz) \\ &= \delta_{mn} \frac{1}{2\pi} \int_{\mathbb{C}} e^{-|\zeta|^2/2} L(d\zeta) = \delta_{mn}. \end{aligned}$$

This completes the proof. □

Set $\phi_n(x) = T_h^* \varphi_n(x)$, $n = 0, 1, 2, \dots$. Since T_h is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $\mathcal{H}_\Phi(\mathbb{C})$, we have the following.

Theorem 2.2. $\{\phi_n\}_{n=0}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R})$.

In what follows we study the family of functions $\{\phi_n\}_{n=0}^\infty$ in detail. Let Λ be a linear operator on $\mathcal{H}_\Phi(\mathbb{C})$ defined by

$$\Lambda U(z) = \frac{2h \operatorname{Im} C}{|B|^2} \frac{dU}{dz}(z), \quad U \in \mathcal{H}_\Phi(\mathbb{C}).$$

Its Hilbert adjoint is

$$\Lambda^* U(z) = zU(z), \quad U \in \mathcal{H}_\Phi(\mathbb{C}).$$

We call Λ and Λ^* annihilation and creation operators on $\mathcal{H}_\Phi(\mathbb{C})$ respectively. Since φ_n is a monomial of degree n , we have for $n = 0, 1, 2, \dots$

$$\varphi_n(z) = \frac{1}{\sqrt{n!}} \left(\frac{B}{\sqrt{2h \operatorname{Im} C}} \right)^n (\Lambda^*)^n \varphi_0(z), \quad (7)$$

$$\left(\Lambda^* \circ \Lambda + \frac{h \operatorname{Im} C}{|B|^2} \right) \varphi_n(z) = \left(\Lambda \circ \Lambda^* - \frac{h \operatorname{Im} C}{|B|^2} \right) \varphi_n(z) = \frac{h \operatorname{Im} C}{|B|^2} (2n+1) \varphi_n(z). \quad (8)$$

We shall pull back these facts by using T_h^* . Set

$$\begin{aligned} P &:= T_h^* \circ \Lambda \circ T_h, & P^* &:= T_h^* \circ \Lambda^* \circ T_h, \\ H &:= P^* \circ P + \frac{h \operatorname{Im} C}{|B|^2} = P \circ P^* - \frac{h \operatorname{Im} C}{|B|^2}. \end{aligned}$$

To state the concrete form of H , we introduce the Weyl pseudodifferential operators. For an appropriate function $a(x, \xi)$ of $(x, \xi) \in \mathbb{R}^2$, its Weyl quantization is defined by

$$\operatorname{Op}_h^W(a)u(x) = \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{i(x-y)\xi/h} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

for $u \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class on \mathbb{R} . Set $D_x = -id/dx$ for short.

Here we give the concrete forms of operators P , P^* and H on \mathbb{R} .

Proposition 2.3. *We have*

$$\begin{aligned} P &= -\frac{1}{B}(hD_x + \bar{C}x), & P^* &= -\frac{1}{B}(hD_x + Cx), \\ H &= \frac{1}{|B|^2} \left\{ h^2 D_x^2 + |C|^2 x^2 + (C + \bar{C}) \left(xhD_x + \frac{h}{2i} \right) \right\} \\ &= \frac{1}{|B|^2} \operatorname{Op}_h^W \left(\xi^2 + |C|^2 x^2 + (C + \bar{C}) x\xi \right). \end{aligned}$$

Proof. We first compute P and P^* . Since $\Lambda \circ T_h = T_h \circ P$, we deduce that for any $u \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \Lambda \circ T_h u(z) &= \frac{2h \operatorname{Im} C}{|B|^2} \frac{d}{dz} C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} u(x) dx \\ &= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \left\{ \frac{2h \operatorname{Im} C}{|B|^2} \frac{i}{h} \frac{\partial \phi}{\partial z}(z, x) \right\} u(x) dx \\ &= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{2i \operatorname{Im} C}{|B|^2} (Az + Bx) u(x) dx \\ &= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{2i \operatorname{Im} C}{|B|^2} \left(-\frac{iB^2}{2 \operatorname{Im} C} z + Bx \right) u(x) dx \end{aligned}$$

$$\begin{aligned}
&= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{1}{B} (Bz + 2i \operatorname{Im} Cx) u(x) dx \\
&= C_\phi h^{-3/4} \int_{\mathbb{R}} \left((hD_x - Cx) e^{i\phi(z,x)/h} \right) \frac{1}{B} u(x) dx \\
&+ C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{2i \operatorname{Im} Cx}{B} u(x) dx \\
&= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{-1}{B} (hD_x + Cx) u(x) dx \\
&+ C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{2i \operatorname{Im} Cx}{B} u(x) dx \\
&= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{-1}{B} (hD_x + Cx - 2i \operatorname{Im} Cx) u(x) dx \\
&= C_\phi h^{-3/4} \int_{\mathbb{R}} e^{i\phi(z,x)/h} \frac{-1}{B} (hD_x + \bar{C}x) u(x) dx,
\end{aligned}$$

which shows that $P = -(hD_x + \bar{C}x)/\bar{B}$. In the same way, we can obtain $P^* = -(hD_x + Cx)/B$, which is certainly the adjoint of P on $L^2(\mathbb{R})$. Next we compute H . Simple computation gives

$$\begin{aligned}
H &= P^* \circ P + \frac{h \operatorname{Im} C}{|B|^2} = \frac{1}{|B|^2} (hD_x + Cx)(hD_x + \bar{C}x) + \frac{h \operatorname{Im} C}{|B|^2} \\
&= \frac{1}{|B|^2} \left\{ h^2 D_x^2 + |C|^2 x^2 + (C + \bar{C}) x h D_x + \frac{h \bar{C}}{i} + h \operatorname{Im} C \right\} \\
&= \frac{1}{|B|^2} \left\{ h^2 D_x^2 + |C|^2 x^2 + (C + \bar{C}) x h D_x + \frac{h \operatorname{Re} C}{i} \right\} \\
&= \frac{1}{|B|^2} \left\{ h^2 D_x^2 + |C|^2 x^2 + (C + \bar{C}) \left(x h D_x + \frac{h}{2i} \right) \right\},
\end{aligned}$$

which completes the proof. \square

By using the pull-back of (7) and (8), we have for $n = 0, 1, 2, \dots$

$$\begin{aligned}
\phi_n(x) &= \frac{1}{\sqrt{n!}} \left(\frac{B}{\sqrt{2h \operatorname{Im} C}} \right)^n (P^*)^n \phi_0(x) \\
&= \frac{1}{\sqrt{n!}} \left(\frac{-1}{\sqrt{2h \operatorname{Im} C}} \right)^n (hD_x + Cx)^n \phi_0(x) \\
&= \frac{1}{\sqrt{n!}} \left(\frac{-1}{\sqrt{2h \operatorname{Im} C}} \right)^n e^{-iCx^2/2h} (hD_x)^n (e^{iCx^2/2h} \phi_0(x)), \\
H\phi_n &= \frac{h \operatorname{Im} C}{|B|^2} (2n+1) \phi_n.
\end{aligned}$$

If we compute the concrete form of ϕ_0 , then we obtain the Rodrigues formula for $\{\phi_n\}_{n=0}^\infty$.

Theorem 2.4. *We have for $n = 0, 1, 2, \dots$*

$$\begin{aligned}
\phi_0(x) &= \left(\frac{\operatorname{Im} C}{\pi h} \right)^{1/4} e^{-i\bar{C}x^2/2h}, \\
\phi_n(x) &= \left(\frac{\operatorname{Im} C}{\pi h} \right)^{1/4} \frac{1}{\sqrt{n!}} \left(\frac{-1}{\sqrt{2h \operatorname{Im} C}} \right)^n e^{-i \operatorname{Re} Cx^2/2h} e^{\operatorname{Im} Cx^2/2h} (hD_x)^n e^{-\operatorname{Im} Cx^2/h}.
\end{aligned}$$

Proof. Recall the definition of ϕ_0 . We have

$$\begin{aligned}\phi_0(x) &= T_h^* \varphi_0(x) = C_\phi h^{-3/4} \int_{\mathbb{C}} e^{-i\overline{\phi(z,x)}/h} \varphi_0(z) e^{-2\Phi(z)/h} L(dz) \\ &= 2^{-1}(\pi h)^{-5/4} (\text{Im } C)^{-3/4} |B|^2 \int_{\mathbb{C}} e^{F_1(z,x)/h} L(dz),\end{aligned}$$

where

$$F_1(z, x) = -i \left\{ \frac{i(\overline{Bz})^2}{4 \text{Im } C} + \overline{Bz} + \frac{\bar{C}x^2}{2} \right\} - \frac{|Bz|^2}{2 \text{Im } C}.$$

Change the variable $\zeta = \xi + i\eta := \overline{Bz}$, $(\xi, \eta) \in \mathbb{R}^2$. We deduce

$$\begin{aligned}\phi_0(x) &= 2^{-1}(\pi h)^{-5/4} (\text{Im } C)^{-3/4} \int_{\mathbb{C}} e^{F_2(\zeta, x)/h} L(d\zeta), \\ F_2(\zeta, x) &= \frac{\zeta^2}{4 \text{Im } C} - i\zeta x - \frac{i\bar{C}x^2}{2} - \frac{|\zeta|^2}{2 \text{Im } C} \\ &= -\frac{\{\xi - i(\eta - 2 \text{Im } Cx)\}^2}{4 \text{Im } C} - \frac{(\eta - \text{Im } Cx)^2}{\text{Im } C} - \frac{i\bar{C}x^2}{2}.\end{aligned}$$

Then we can obtain

$$\begin{aligned}\phi_0(x) &= 2^{-1}(\pi h)^{-5/4} (\text{Im } C)^{-3/4} e^{-i\bar{C}x^2/2h} \int_{\mathbb{R}} e^{-\xi^2/4h \text{Im } C} d\xi \int_{\mathbb{R}} e^{-\eta^2/h \text{Im } C} d\eta \\ &= \left(\frac{\text{Im } C}{\pi h} \right)^{1/4} e^{-i\bar{C}x^2/2h}.\end{aligned}$$

This completes the proof. \square

3. THE COMMUTATIVE CASE OF NON-COMMUTATIVE HARMONIC OSCILLATORS

Consider a 2×2 system of second-order differential operators of the form

$$Q_{(\alpha, \beta)} = \frac{1}{2} A_{(\alpha, \beta)} \text{Op}_h^W(\xi^2 + x^2) + J \text{Op}_h^W(ix\xi),$$

where α and β are positive constants satisfying $\alpha\beta > 1$, and

$$A_{(\alpha, \beta)} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

A matrix $A_{(\alpha, \beta)}(\xi^2 + x^2)/2 + J(ix\xi)$, which is the symbol of the operator $Q_{(\alpha, \beta)}$, is a Hermitian matrix, and all its eigenvalues are real-valued. Note that all its eigenvalues are positive for $(x, \xi) \neq (0, 0)$ if and only if $\alpha\beta > 1$. In other words, $Q_{(\alpha, \beta)}$ is a system of semiclassical elliptic differential operators if and only if $\alpha\beta > 1$. The system of differential operators $Q_{(\alpha, \beta)}$ was mathematically introduced in [6] by Parmeggiani and Wakayama. They call $Q_{(\alpha, \beta)}$ a Hamiltonian of non-commutative harmonic oscillator. The word “non-commutative” comes from the non-commutativity $A_{(\alpha, \beta)}J \neq JA_{(\alpha, \beta)}$ for $\alpha \neq \beta$. It is not known that the system of differential equations for $Q_{(\alpha, \beta)}$ describes a physical phenomenon.

Parmeggiani and Wakayama intensively studied spectral properties of $Q_{(\alpha, \beta)}$ in [6], [7] and [8]. See also a monograph [5]. They proved that if $\alpha\beta > 1$, then $Q_{(\alpha, \beta)}$ is a self-adjoint and positive operator, and its spectra consists of positive eigenvalues whose multiplicities are at most three. In case of $\alpha = \beta$, they obtained more detail.

The purpose of the present section is to give alternative proof of the results of Parmeggiani and Wakayama for the commutative case $\alpha = \beta$. More precisely, we study $Q_{(\alpha, \alpha)}$ by using the results in the previous section.

In what follows we assume that $\alpha = \beta$. Then $\alpha > 1$ since $\alpha > 0$ and $\alpha^2 = \alpha\beta > 1$. Let I be the 2×2 identity matrix. Set $Q_\alpha = Q_{(\alpha, \alpha)}$ for short. Let U be a 2×2 unitary matrix defined by

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

which diagonalize J as

$$UJU^* = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

Then, we have

$$\begin{aligned} UQ_\alpha U^* &= \frac{\alpha}{2} I \operatorname{Op}_h^W(\xi^2 + x^2) + iUJU^* \operatorname{Op}_h^W(x\xi) = \begin{bmatrix} H_{\alpha,+} & 0 \\ 0 & H_{\alpha,-} \end{bmatrix}, \\ H_{\alpha,\pm} &= \operatorname{Op}_h^W(\xi^2 + x^2) \pm \operatorname{Op}_h^W(x\xi) = \operatorname{Op}_h^W \left(\left| \sqrt{\frac{\alpha}{2}} (\nu_{\alpha,\pm} \xi + x) \right|^2 \right), \\ \nu_{\alpha,\pm} &= \frac{\pm 1 + i\sqrt{\alpha^2 - 1}}{\alpha}. \end{aligned}$$

Note that $|\nu_{\alpha,\pm}| = 1$ and $\operatorname{Im} \nu_{\alpha,\pm} > 0$.

Here we make use of the results in the previous section by setting

$$B = \sqrt{\frac{2}{\alpha}} \nu_{\alpha,\pm}, \quad C = \nu_{\alpha,\pm}, \quad A = \frac{B^2}{2i \operatorname{Im} C} = \frac{\nu_{\alpha,\pm}^2}{i\sqrt{\alpha^2 - 1}}.$$

Note that the requirement $\operatorname{Im} C > 0$ is satisfied. Set

$$\begin{aligned} \phi_{\alpha,\pm,n}(x) &= e^{\mp ix^2/2\alpha h} h_{\alpha,n}(x), \\ h_{\alpha,n}(x) &= \left(\frac{\sqrt{\alpha^2 - 1}}{\pi h \alpha} \right)^{1/4} \frac{1}{\sqrt{n!}} \left(-\sqrt{\frac{\alpha}{2\sqrt{\alpha^2 - 1}h}} \right)^n \\ &\quad \times e^{\sqrt{\alpha^2 - 1}x^2/2\alpha h} (hD_x)^n e^{-\sqrt{\alpha^2 - 1}x^2/\alpha h} \end{aligned}$$

for $n = 0, 1, 2, \dots$. Then we deduce that $\{\phi_{\alpha,\pm,n}\}_{n=0}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R})$, and

$$H_{\alpha,\pm} \phi_{\alpha,\pm,n} = \frac{\sqrt{\alpha^2 - 1}}{2} h(2n + 1) \phi_{\alpha,\pm,n}, \quad n = 0, 1, 2, \dots$$

In order to get the eigenfunctions of Q_α , we set

$$\Phi_{\alpha,+,n}(x) = U^* \begin{bmatrix} \phi_{\alpha,+,n} \\ 0 \end{bmatrix}, \quad \Phi_{\alpha,-,n}(x) = U^* \begin{bmatrix} 0 \\ \phi_{\alpha,-,n} \end{bmatrix}, \quad n = 0, 1, 2, \dots,$$

those are,

$$\Phi_{\alpha,\pm,n}(x) = h_{\alpha,n}(x) \cdot \frac{e^{\mp ix^2/2\alpha h}}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}, \quad n = 0, 1, 2, \dots$$

We have proved the results of this section as follows.

Theorem 3.1. *A system of \mathbb{C}^2 -valued functions $\{\Phi_{\alpha,\mu,n} \mid \mu = \pm, n = 0, 1, 2, \dots\}$ is a complete orthonormal system of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, and satisfies*

$$Q_\alpha \Phi_{\alpha,\pm,n} = \frac{\sqrt{\alpha^2 - 1}}{2} h(2n + 1) \Phi_{\alpha,\pm,n}, \quad n = 0, 1, 2, \dots$$

This is not a new result. This was first proved by Parmeggiani and Wakayama in [6]. We believe that our method of proof is easier than that of [6].

4. ORTHOGONAL SYSTEMS ASSOCIATED WITH ELLIPSES IN THE PHASE PLANE

Throughout of the present section, we assume that $h = 1$ for the sake of simplicity. We begin with recalling the relationship between the standard Bargmann transform B_1 and circles in the phase plane. Here we introduce a Berezin-Toeplitz quantization on \mathbb{C} . Let $b(z)$ be an appropriate function on \mathbb{C} . Set

$$a(x, \xi) = \frac{1}{\pi} \iint_{\mathbb{R}^2} e^{-(x-y)^2 - (\xi-\eta)^2} b(y - i\eta) dy d\eta.$$

It is known that

$$(\text{Op}_1^W(a)u, v)_{L^2} = (bB_1u, B_1v)_{L_B^2} = (\tilde{T}_b B_1u, B_1v)_{\mathcal{H}_B}$$

for $u, v \in \mathcal{S}(\mathbb{R})$. See, e.g., [9] and [2]. The operator \tilde{T}_b is said to be the Berezin-Toeplitz quantization of b , which acts on $\mathcal{H}_B(\mathbb{C})$. If b is a characteristic function on \mathbb{C} , then $\text{Op}_1^W(a)$ is said to be a Daubechies' localization operator introduced in [3]. Moreover Daubechies proved that if $b(z)$ is radially symmetric, that is, b is of the form $b(x - i\xi) = c(x^2 + \xi^2)$ with some function $c(s)$ for $s \geq 0$, then all the usual Hermite functions $\phi_{B,n}$ ($n = 0, 1, 2, \dots$) are the eigenfunctions of \tilde{T}_b :

$$\tilde{T}_b \phi_{B,n} = \lambda_n \phi_{B,n}, \quad \lambda_n = \frac{1}{n!} \int_0^\infty c(2s) s^n e^{-s} ds, \quad n = 0, 1, 2, \dots$$

Recently, Daubechies' results have been developed. Here we quote two interesting results of inverse problems studied in [1] and [10]. On one hand, in [10] Yoshino proved that radially symmetric symbols of the Berezin-Toeplitz quantization on \mathbb{C} can be reconstructed by all the eigenvalues $\{\lambda_n\}_{n=0}^\infty$. More precisely, by using the framework of hyperfunctions, he obtained the reconstruction formula for radially symmetric symbols. On the other hand, in [1] Abreu and Dörfler studied the inverse problem for Daubechies' localization operators. Let Ω be a bounded subset of \mathbb{C} , and let $b(z)$ be the characteristic function of Ω . They proved that if there exists a nonnegative integer m such that the m -th Hermite function $\phi_{B,m}$ is an eigenfunction of $\text{Op}_1^W(a)$, then Ω must be a disk centered at the origin. In this case it follows automatically that all the Hermite functions $\phi_{B,n}$ are eigenfunctions of $\text{Op}_1^W(a)$ associated with eigenvalues

$$\lambda_n = e^{-R} \sum_{k=n+1}^\infty \frac{R^k}{k!}, \quad n = 0, 1, 2, \dots,$$

respectively, where R is the radius of Ω . In particular $R = -\log(1 - \lambda_0)$. That is the review of the relationship between the usual Bargmann transform and circles (or disks) in \mathbb{C} .

The purpose of the present section is to consider the possibility of the extension of the above to ellipses (or elliptic disks) in \mathbb{C} . Unfortunately, however, we could not obtain the extension of the above. In what follows we introduce a family of functions which might be concerned with ellipses in \mathbb{C} .

Let $\alpha > 0$ and let $\beta \in \mathbb{R}$. Suppose that $(\alpha, \beta) \neq (1, 0)$. For $\rho > 0$,

$$E_\rho := \{x - i\xi \in \mathbb{C} \mid (x, \xi) \in \mathbb{R}^2, |\alpha x - i(\beta x + \xi)| \leq \rho\}$$

is an elliptic disk in \mathbb{C} . Note that E_ρ is a usual disk if and only if $(\alpha, \beta) = (1, 0)$. Note that

$$\{\partial E_\rho \mid \alpha > 0, \beta \in \mathbb{R}, \rho > 0\}$$

is the set of all ellipses centered at the origin, where $\partial E_\rho = \{x - i\xi \in \mathbb{C} \mid |\alpha x - i(\beta x + \xi)| = \rho\}$. Indeed, consider a function

$$F(x, y; a, b, \theta) := a(x \cos \theta - \xi \sin \theta)^2 + b(x \sin \theta + \xi \cos \theta)^2, \quad a > 0, b > 0, \theta \in [0, 2\pi].$$

Elementary computation gives

$$\frac{F(x, y; a, b, \theta)}{a \sin^2 \theta + b \cos^2 \theta} = \frac{ab}{(a \sin^2 \theta + b \cos^2 \theta)^2} x^2 + \left\{ \frac{(b-a) \sin \theta \cos \theta}{a \sin^2 \theta + b \cos^2 \theta} x + \xi \right\}^2.$$

Here we introduce a function $\psi_0(z)$ which seems to be related with an elliptic disk E_ρ . Set $z = x - i\xi$ and $\zeta = \alpha x - i(\beta x + \xi)$ for $(x, \xi) \in \mathbb{R}^2$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then

$$\zeta = \frac{\alpha + 1 - i\beta}{2} z + \frac{\alpha - 1 - i\beta}{2} \bar{z}, \quad z = \frac{\alpha + 1 + i\beta}{2\alpha} \zeta - \frac{\alpha - 1 - i\beta}{2\alpha} \bar{\zeta}.$$

We define the function $\psi_0(z)$ by

$$\psi_0(z) = \exp\left(-\frac{a}{4} z^2\right), \quad a = \frac{\alpha^2 + \beta^2 - 1 + 2i\beta}{\alpha^2 + \beta^2 + 1}.$$

Let $\|\cdot\|_{\mathcal{H}_B}$ be the norm of $\mathcal{H}_B(\mathbb{C})$ determined by the inner product $(\cdot, \cdot)_{\mathcal{H}_B}$. The properties of $\psi_0(z)$ are the following.

Lemma 4.1. *We have*

- (i) $\psi_0 \in \mathcal{H}_B(\mathbb{C})$.
- (ii) $|\psi_0(z)|^2 e^{-|z|^2/2} = e^{-|\zeta|^2/(\alpha^2 + \beta^2 + 1)}$.
- (iii) $\|\psi_0\|_{\mathcal{H}_B}^2 = (\alpha^2 + \beta^2 + 1)\pi/\alpha$.

Proof. We first show (i). We have only to show the integrability of $|\psi_0(z)|^2 e^{-|z|^2/2}$ since $\psi_0(z)$ is an entire function. Note that

$$|a|^2 = \frac{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2}{(\alpha^2 + \beta^2 + 1)^2} = 1 - \left(\frac{2\alpha}{\alpha^2 + \beta^2 + 1}\right)^2.$$

We have

$$0 < \frac{2\alpha}{\alpha^2 + \beta^2 + 1} < 1$$

since

$$\alpha^2 + \beta^2 + 1 - 2\alpha = (\alpha - 1)^2 + \beta^2 > 0$$

for $(\alpha, \beta) \neq (1, 0)$. Thus $0 < |a| < 1$. We deduce that

$$\begin{aligned} |\psi_0(z)|^2 e^{-|z|^2/2} &\leq \exp\left(-\frac{a}{4} z^2 - \frac{\bar{a}}{4} \bar{z}^2 - \frac{1}{2} |z|^2\right) \leq \exp\left(-\frac{1 - |a|}{2} |z|^2\right) \\ &= \exp\left\{-\frac{1}{2} \left(1 - \sqrt{1 - \left(\frac{2\alpha}{\alpha^2 + \beta^2 + 1}\right)^2}\right) |z|^2\right\}. \end{aligned}$$

This implies that $|\psi_0(z)|^2 e^{-|z|^2/2}$ is integrable on \mathbb{C} with respect to the Lebesgue measure $L(dz)$ and $\psi_0 \in \mathcal{H}_B(\mathbb{C})$.

We show (ii) and (iii). Elementary computation shows that

$$\frac{|\zeta|^2}{\alpha^2 + \beta^2 + 1} = \frac{|z|^2}{2} + \frac{a}{4} z^2 + \frac{\bar{a}}{4} \bar{z}^2,$$

which implies (ii). Moreover, it is easy to see that $dz \wedge d\bar{z} = \alpha^{-1} d\zeta \wedge d\bar{\zeta}$ and

$$\|\psi_0\|_{\mathcal{H}_B}^2 = \int_{\mathbb{C}} |\psi_0(z)|^2 e^{-|z|^2/2} L(dz) = \frac{1}{\alpha} \int_{\mathbb{C}} e^{-|\zeta|^2/(\alpha^2 + \beta^2 + 1)} L(d\zeta) = \frac{\alpha^2 + \beta^2 + 1}{\alpha} \pi.$$

This completes the proof. \square

The identity $|\psi_0(z)|^2 e^{-|z|^2/2} = e^{-|\zeta|^2/(\alpha^2+\beta^2+1)}$ makes us to expect that ψ_0 might be related with an elliptic disk E_ρ and generate a family of eigenfunctions for the Daubechies' localization operators supported in E_ρ . Unfortunately, however, this expectation fails to hold. The purpose of the present section is to generate a family of functions by ψ_0 , and show its properties similar to the previous sections. To state our results in the present section, we here introduce some notation. Set

$$\lambda = \frac{2\alpha^2}{(\alpha^2 + \beta^2 + 1)(\alpha^2 + \beta^2 - 1 - 2i\beta)}, \quad \Lambda_{\alpha,\beta} = \frac{1}{a} \frac{\partial}{\partial z} + \frac{z}{2}, \quad \Lambda_{\alpha,\beta}^* = \frac{\partial}{\partial z} + \frac{a + 2\lambda}{2} z.$$

It is easy to see that $a + 2\lambda = 1/\bar{a}$, $\Lambda_{\alpha,\beta}\psi_0 = 0$, and $\Lambda_{\alpha,\beta}^*$ is the Hilbert adjoint of $\Lambda_{\alpha,\beta}$ on $\mathcal{H}_B(\mathbb{C})$. We make use of ψ_0 , $\Lambda_{\alpha,\beta}$ and $\Lambda_{\alpha,\beta}^*$ as a generating element of a family of functions, and annihilation and creation operators respectively. Set $\psi_n = (\Lambda_{\alpha,\beta}^*)^n \psi_0$ for $n = 0, 1, 2, \dots$, and set

$$C_{\alpha,\beta} = \frac{1 - \bar{a}}{2\bar{a}} = \frac{1 + i\beta}{\alpha^2 + \beta^2 - 1 - 2i\beta}$$

for short. Properties of $\Lambda_{\alpha,\beta}$, $\Lambda_{\alpha,\beta}^*$ and $\{\psi_n\}_{n=0}^\infty$ are the following.

Theorem 4.2.

(i) $\{\psi_n\}_{n=0}^\infty$ satisfies a formula of the form

$$\psi_n(z) = \left\{ e^{-\lambda z^2/2} \left(\frac{\partial}{\partial z} \right)^n e^{\lambda z^2/2} \right\} \psi_0(z), \quad n = 0, 1, 2, \dots$$

(ii) For $n = 1, 2, 3, \dots$,

$$\Lambda_{\alpha,\beta}(\Lambda_{\alpha,\beta}^*)^n = (\Lambda_{\alpha,\beta}^*)^n \Lambda_{\alpha,\beta} + n \frac{\lambda}{a} (\Lambda_{\alpha,\beta}^*)^{n-1}.$$

(iii) $\{\psi_n\}_{n=0}^\infty$ is a complete orthogonal system of $\mathcal{H}_B(\mathbb{C})$.

(iv) For $n = 0, 1, 2, \dots$,

$$\left(\frac{\Lambda_{\alpha,\beta}^* \Lambda_{\alpha,\beta}}{|C_{\alpha,\beta}|^2} + \frac{\alpha^2}{1 + \beta^2} \right) \psi_n = \frac{\alpha^2}{1 + \beta^2} (2n + 1) \psi_n.$$

Proof. First we show (i). Note that for any $c \in \mathbb{C}$ and for any holomorphic function $f(z)$, we deduce that

$$\begin{aligned} \left(\frac{\partial}{\partial z} + cz \right) f(z) &= \left(\frac{\partial}{\partial z} + cz \right) \{ e^{-cz^2/2} (e^{cz^2/2} f(z)) \} \\ &= -c z e^{-cz^2/2} (e^{cz^2/2} f(z)) + e^{-cz^2/2} \frac{\partial}{\partial z} (e^{cz^2/2} f(z)) + c z e^{-cz^2/2} (e^{cz^2/2} f(z)) \\ &= e^{-cz^2/2} \frac{\partial}{\partial z} (e^{cz^2/2} f(z)). \end{aligned}$$

Using this repeatedly, we have

$$\begin{aligned} \psi_n(z) &= (\Lambda_{\alpha,\beta})^n \psi_0(z) = \left\{ \frac{\partial}{\partial z} + \left(\frac{a}{2} + \lambda \right) \right\}^n e^{-az^2/4} \\ &= e^{-az^2/4 - \lambda z^2/2} \left(\frac{\partial}{\partial z} \right)^n (e^{az^2/4 + \lambda z^2/2} \cdot e^{-az^2/4}) = e^{-az^2/4 - \lambda z^2/2} \left(\frac{\partial}{\partial z} \right)^n e^{\lambda z^2/2} \\ &= \left\{ e^{-\lambda z^2/2} \left(\frac{\partial}{\partial z} \right)^n e^{\lambda z^2/2} \right\} e^{-az^2/4} = \left\{ e^{-\lambda z^2/2} \left(\frac{\partial}{\partial z} \right)^n e^{\lambda z^2/2} \right\} \psi_0(z), \end{aligned}$$

which is desired.

Next we show (ii). We employ induction on n . For $n = 1$, we deduce that

$$\Lambda_{\alpha,\beta} \Lambda_{\alpha,\beta}^* - \Lambda_{\alpha,\beta}^* \Lambda_{\alpha,\beta} = \frac{1}{a} \left(\frac{\partial}{\partial z} + \frac{a}{2} z \right) \left\{ \frac{\partial}{\partial z} + \left(\frac{a}{2} + \lambda \right) z \right\}$$

$$\begin{aligned}
& - \left\{ \frac{\partial}{\partial z} + \left(\frac{a}{2} + \lambda \right) z \right\} \frac{1}{a} \left(\frac{\partial}{\partial z} + \frac{a}{2} z \right) \\
& = \frac{1}{a} \left(\frac{\partial}{\partial z} + \frac{a}{2} z \right) (\lambda z) - (\lambda z) \frac{1}{a} \left(\frac{\partial}{\partial z} + \frac{a}{2} z \right) (\lambda z) \\
& = \frac{\lambda}{a} = 1 \cdot \frac{\lambda}{a} \cdot (\Lambda_{\alpha, \beta}^*)^0.
\end{aligned}$$

Here we suppose that (ii) holds for some $n = 1, 2, 3, \dots$. We show the case of $n + 1$. By using the cases of n and 1, we deduce that

$$\begin{aligned}
\Lambda_{\alpha, \beta}(\Lambda_{\alpha, \beta}^*)^{n+1} &= \Lambda_{\alpha, \beta}(\Lambda_{\alpha, \beta}^*)^n \Lambda_{\alpha, \beta}^* \\
&= \left\{ (\Lambda_{\alpha, \beta}^*)^n \Lambda_{\alpha, \beta} + n \frac{\lambda}{a} (\Lambda_{\alpha, \beta}^*)^{n-1} \right\} \Lambda_{\alpha, \beta}^* \\
&= (\Lambda_{\alpha, \beta}^*)^n (\Lambda_{\alpha, \beta} \Lambda_{\alpha, \beta}^*) + n \frac{\lambda}{a} (\Lambda_{\alpha, \beta}^*)^n \\
&= (\Lambda_{\alpha, \beta}^*)^n \left\{ \Lambda_{\alpha, \beta}^* \Lambda_{\alpha, \beta} + \frac{\lambda}{a} \right\} + n \frac{\lambda}{a} (\Lambda_{\alpha, \beta}^*)^n \\
&= (\Lambda_{\alpha, \beta}^*)^{n+1} \Lambda_{\alpha, \beta} + (n+1) \frac{\lambda}{a} (\Lambda_{\alpha, \beta}^*)^n,
\end{aligned}$$

which is desired.

For (iii), we here show only the orthogonality

$$\begin{aligned}
(\psi_m, \psi_n)_{\mathcal{H}_B} &= \left(\frac{\lambda}{a} \right)^n n! \|\psi_0\|_{\mathcal{H}_B}^2 \delta_{mn} \\
&= \frac{(\alpha^2 + \beta^2 + 1)\pi}{\alpha} \left(\frac{2\alpha^2}{(\alpha^2 + \beta^2 - 1)^2 + 4\beta^2} \right)^n n! \delta_{mn}, \quad m, n = 0, 1, 2, \dots \quad (9)
\end{aligned}$$

The completeness will be automatically proved later. Recall that $\Lambda_{\alpha, \beta} \psi_0 = 0$. Suppose that $m \geq n$. By using (ii) repeatedly, we deduce that

$$\begin{aligned}
(\psi_m, \psi_n)_{\mathcal{H}_B} &= ((\Lambda_{\alpha, \beta}^*)^m \psi_0, (\Lambda_{\alpha, \beta}^*)^n \psi_0)_{\mathcal{H}_B} \\
&= ((\Lambda_{\alpha, \beta}^*)^{m-1} \psi_0, \Lambda_{\alpha, \beta} (\Lambda_{\alpha, \beta}^*)^n \psi_0)_{\mathcal{H}_B} \\
&= \left((\Lambda_{\alpha, \beta}^*)^{m-1} \psi_0, (\Lambda_{\alpha, \beta}^*)^n \Lambda_{\alpha, \beta} \psi_0 + n \frac{\lambda}{a} (\Lambda_{\alpha, \beta}^*)^{n-1} \psi_0 \right)_{\mathcal{H}_B} \\
&= n \frac{\lambda}{a} ((\Lambda_{\alpha, \beta}^*)^{m-1} \psi_0, (\Lambda_{\alpha, \beta}^*)^{n-1} \psi_0)_{\mathcal{H}_B} \\
&= \dots \\
&= n! \left(\frac{\lambda}{a} \right)^n ((\Lambda_{\alpha, \beta}^*)^{m-n} \psi_0, \psi_0)_{\mathcal{H}_B}.
\end{aligned}$$

If $m > n$, then

$$(\psi_m, \psi_n)_{\mathcal{H}_B} = n! \left(\frac{\lambda}{a} \right)^n ((\Lambda_{\alpha, \beta}^*)^{m-n-1} \psi_0, \Lambda_{\alpha, \beta} \psi_0)_{\mathcal{H}_B} = 0.$$

If $m = n$, then

$$(\psi_n, \psi_n)_{\mathcal{H}_B} = n! \left(\frac{\lambda}{a} \right)^n (\psi_0, \psi_0)_{\mathcal{H}_B} = n! \left(\frac{\lambda}{a} \right)^n \|\psi_0\|_{\mathcal{H}_B}^2.$$

Finally we show (iv). By using (ii) and $\Lambda_{\alpha,\beta}\psi_0 = 0$ again, we deduce that

$$\begin{aligned} \left(\frac{\Lambda_{\alpha,\beta}^* \Lambda_{\alpha,\beta}}{|C_{\alpha,\beta}|^2} + \frac{\alpha^2}{1+\beta^2} \right) \psi_n &= \frac{\Lambda_{\alpha,\beta}^* \Lambda_{\alpha,\beta} (\Lambda_{\alpha,\beta}^*)^n}{|C_{\alpha,\beta}|^2} \psi_0 + \frac{\alpha^2}{1+\beta^2} \psi_n \\ &= \frac{1}{|C_{\alpha,\beta}|^2} \left\{ (\Lambda_{\alpha,\beta}^*)^{n+1} \Lambda_{\alpha,\beta} + n \frac{\lambda}{a} (\Lambda_{\alpha,\beta}^*)^n \right\} \psi_0 + \frac{\alpha^2}{1+\beta^2} \psi_n \\ &= \frac{n\lambda}{a|C_{\alpha,\beta}|^2} (\Lambda_{\alpha,\beta}^*)^n \psi_0 + \frac{\alpha^2}{1+\beta^2} \psi_n \\ &= \left\{ \frac{n\lambda}{a|C_{\alpha,\beta}|^2} + \frac{\alpha^2}{1+\beta^2} \right\} \psi_n = \frac{\alpha^2}{1+\beta^2} (2n+1) \psi_n. \end{aligned}$$

This completes the proof. \square

Here we introduce a family of functions $\{\Psi_n\}_{n=0}^\infty$ on \mathbb{R} by setting $\Psi_n = B_1^* \psi_n$ for $n = 0, 1, 2, \dots$. In order to study basic properties on $\{\Psi_n\}_{n=0}^\infty$, we introduce notation. Set

$$P_{\alpha,\beta} := \frac{1}{C_{\alpha,\beta}} B_1^* \circ \Lambda_{\alpha,\beta} \circ B_1, \quad P_{\alpha,\beta}^* := \frac{1}{C_{\alpha,\beta}} B_1^* \circ \Lambda_{\alpha,\beta}^* \circ B_1, \quad H_{\alpha,\beta} := P_{\alpha,\beta}^* \circ P_{\alpha,\beta} + \frac{\alpha^2}{1+\beta^2}$$

for short. Then we have

$$\begin{aligned} P_{\alpha,\beta} &= \frac{d}{dx} + \frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{1+\beta^2} x, \quad P_{\alpha,\beta}^* = -\frac{d}{dx} + \frac{\alpha^2 - i(\alpha^2 + \beta^2 + 1)\beta}{1+\beta^2} x, \\ H_{\alpha,\beta} &= -\frac{d^2}{dx^2} + \frac{\alpha^4 + (\alpha^2 + \beta^2 + 1)\beta^2}{(1+\beta^2)^2} x^2 - 2i \frac{(\alpha^2 + \beta^2 + 1)\beta}{1+\beta^2} x \frac{d}{dx} - i \frac{(\alpha^2 + \beta^2 + 1)\beta}{1+\beta^2} \\ &= \text{Op}^w \left(\xi^2 + \frac{\alpha^4 + (\alpha^2 + \beta^2 + 1)\beta^2}{(1+\beta^2)^2} x^2 + 2 \frac{(\alpha^2 + \beta^2 + 1)\beta}{1+\beta^2} x \xi \right). \end{aligned}$$

Set

$$A_{\alpha,\beta} = \pi^{1/4} \left\{ \frac{\alpha^2 + \beta^2 + 1}{1 - i\beta} \right\}^{1/2},$$

where we take its argument in $(-\pi/4, \pi/4)$. Our results in the present section are the following.

Theorem 4.3.

(i) We have for $n = 0, 1, 2, \dots$

$$\begin{aligned} \Psi_n(x) &= A_{\alpha,\beta} (-C_{\alpha,\beta})^n \exp \left(-i \frac{(\alpha^2 + \beta^2 + 1)\beta}{2(1+\beta^2)} x^2 \right) \\ &\quad \times \exp \left(\frac{\alpha^2}{2(1+\beta^2)} x^2 \right) \left(\frac{d}{dx} \right)^n \exp \left(-\frac{\alpha^2}{1+\beta^2} x^2 \right). \end{aligned}$$

In particular,

$$\Psi_0(x) = A_{\alpha,\beta} \exp \left(-\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1+\beta^2)} x^2 \right).$$

(ii) $\{\Psi_n\}_{n=0}^\infty$ is a family of eigenfunctions of $H_{\alpha,\beta}$, that is,

$$H_{\alpha,\beta} \Psi_n = \frac{\alpha^2}{1+\beta^2} (2n+1) \Psi_n, \quad n = 0, 1, 2, \dots$$

(iii) $\{\Psi_n\}_{n=0}^\infty$ is a complete orthogonal system of $L^2(\mathbb{R})$.

Recall that Theorem 4.2 was proved except for the completeness of $\{\psi_n\}_{n=0}^\infty$. Theorem 4.2 without the completeness implies (ii) of Theorem 4.3 and the orthogonality of $\{\Psi_n\}_{n=0}^\infty$ in $L^2(\mathbb{R})$. If (i) of Theorem 4.3 holds, then the completeness of $\{\Psi_n\}_{n=0}^\infty$ in $L^2(\mathbb{R})$ follows immediately. Indeed, combining (i) of Theorem 4.3 and the results in Section 2 with

$$A = \frac{i(1 + \beta^2)}{2\alpha^2}, \quad B = -i, \quad C = \frac{(\alpha^2 + \beta^2 + 1)\beta + i\alpha^2}{1 + \beta^2},$$

we can check the completeness of $\{\Psi_n\}_{n=0}^\infty$ in $L^2(\mathbb{R})$. This implies the completeness of $\{\psi_n\}_{n=0}^\infty$ in $\mathcal{H}_B(\mathbb{C})$ stated in Theorem 4.2 since B_1 is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $\mathcal{H}_B(\mathbb{C})$. For this reason, we have only to show (i) of Theorem 4.3. For this purpose, we need the following.

Lemma 4.4. *Let $\rho > 0$ and let $2\theta \in (-\pi/2, \pi/2)$. Then we have*

$$\int_{\mathbb{R}} \exp(-\rho^2 e^{2i\theta} t^2) dt = \frac{\sqrt{\pi}}{\rho e^{i\theta}}.$$

Proof. The integrand is an even function of $t \in \mathbb{R}$. By using change of variable $t \mapsto \rho t$, we have

$$\int_{\mathbb{R}} \exp(-\rho^2 e^{2i\theta} t^2) dt = \frac{2}{\rho} \int_0^\infty \exp(-e^{2i\theta} t^2) dt.$$

Let $R > 0$. Consider a contour γ_R which consists of $\gamma_R = \gamma_R(1) \cup \gamma_R(2) \cup \gamma_R(3)$, where

$$\gamma_R(1) = \{t \mid t \in [0, R]\}, \quad \gamma_R(3) = \{e^{i\theta}(R - t) \mid t \in [0, R]\},$$

$$\gamma_R(2) = \{Re^{it} \mid t \in [0, \theta]\} \quad (\theta \geq 0), \quad \gamma_R(2) = \{Re^{i(\theta-t)} \mid t \in [\theta, 0]\} \quad (\theta < 0).$$

Applying Cauchy's theorem to the holomorphic function e^{-z^2} on γ_R , we have

$$e^{i\theta} \int_0^R \exp(-e^{2i\theta} t^2) dt = \int_0^R e^{-t^2} dt + iR \int_0^\theta \exp(-R^2 e^{2it} + it) dt.$$

Here we note that $0 < \cos(2\theta) \leq 1$ since $2\theta \in (-\pi/2, \pi/2)$. Then we deduce that

$$\begin{aligned} \left| \exp(-R^2 e^{2it} + it) \right| &= \exp(-R^2 \cos(2t)) \leq \exp(-R^2 \cos(2\theta)) \\ &= \frac{1}{\sum_{k=0}^\infty \frac{(R^2 \cos(2\theta))^k}{k!}} \leq \frac{1}{R^2 \cos(2\theta)}, \end{aligned}$$

and

$$\begin{aligned} \left| iR \int_0^\theta \exp(-R^2 e^{2it} + it) dt \right| &\leq R \left| \int_0^\theta \exp(-R^2 e^{2it} + it) dt \right| \\ &\leq R \left| \int_0^\theta \frac{1}{R^2 \cos(2\theta)} dt \right| = \frac{|\theta|}{R \cos(2\theta)}. \end{aligned}$$

Then we have

$$\int_0^R \exp(-e^{2i\theta} t^2) dt = \frac{1}{e^{i\theta}} \int_0^R e^{-t^2} dt + \mathcal{O}\left(\frac{1}{R}\right) \quad (R \rightarrow \infty),$$

and

$$\int_0^\infty \exp(-e^{2i\theta} t^2) dt = \frac{1}{e^{i\theta}} \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2e^{i\theta}}.$$

This completes the proof. □

Finally we complete the proof of Theorem 4.3.

Proof of Theorem 4.3. It suffices to show the part (i). We first compute the concrete form of $\Psi_0(x)$. Recall the definition of $\Psi_0(x)$

$$\Psi_0(x) = B_1^* \psi_0(x) = 2^{-1/2} \pi^{-3/4} \iint_{\mathbb{R}^2} e^{G(y, \eta, x)} dy d\eta,$$

where

$$G(y, \eta, x) = -\frac{(y + i\eta)^2}{4} + (y + i\eta)x - \frac{x^2}{2} - \frac{a(y - i\eta)^2}{4} - \frac{y^2 + \eta^2}{2}.$$

Elementary computation gives

$$\begin{aligned} G(y, \eta, x) = & -\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \\ & - \frac{2\alpha^2 + 2\beta^2 + 1 + i\beta}{2(\alpha^2 + \beta^2 + 1)} (y - z_1)^2 - \frac{1 - i\beta}{2\alpha^2 + 2\beta^2 + 1 + i\beta} (\eta - z_2)^2, \end{aligned}$$

where

$$z_1 = \frac{2x - i(1 - a)\eta}{3 + a}, \quad z_2 = i \frac{\alpha^2 + \beta^2 + i\beta}{1 - i\beta} x.$$

By using this and Lemma 4.4, we deduce that

$$\begin{aligned} \Psi_0(x) = & 2^{-1/2} \pi^{-3/4} \exp \left(-\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \right) \\ & \times \int_{\mathbb{R}} \exp \left(-\frac{1 - i\beta}{2\alpha^2 + 2\beta^2 + 1 + i\beta} (\eta - z_2)^2 \right) d\eta \\ & \times \int_{\mathbb{R}} \exp \left(-\frac{2\alpha^2 + 2\beta^2 + 1 + i\beta}{2(\alpha^2 + \beta^2 + 1)} (y - z_1)^2 \right) dy \\ = & 2^{-1/2} \pi^{-3/4} \exp \left(-\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \right) \\ & \times \int_{\mathbb{R}} \exp \left(-\frac{1 - i\beta}{2\alpha^2 + 2\beta^2 + 1 + i\beta} \eta^2 \right) d\eta \\ & \times \int_{\mathbb{R}} \exp \left(-\frac{2\alpha^2 + 2\beta^2 + 1 + i\beta}{2(\alpha^2 + \beta^2 + 1)} y^2 \right) dy \\ = & 2^{-1/2} \pi^{-3/4} \times \left\{ \frac{1 - i\beta}{2\alpha^2 + 2\beta^2 + 1 + i\beta} \cdot \frac{2\alpha^2 + 2\beta^2 + 1 + i\beta}{2(\alpha^2 + \beta^2 + 1)} \right\}^{-1/2} \pi \\ & \times \exp \left(-\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \right) \\ = & A_{\alpha, \beta} \exp \left(-\frac{\alpha^2 + i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \right), \end{aligned}$$

which is desired.

Finally we check the Rodrigues formula for Ψ_n . By using the definition of Ψ_n , we deduce that

$$\begin{aligned} \Psi_n(x) = & B^*(\Lambda_{\alpha, \beta}^*)^n \psi_0(x) = (C_{\alpha, \beta} P_{\alpha, \beta}^*)^n B^* \psi_0(x) = (C_{\alpha, \beta})^n (P_{\alpha, \beta}^*)^n \Psi_0(x) \\ = & (-C_{\alpha, \beta})^n \left(\frac{d}{dx} - \frac{\alpha^2 - i(\alpha^2 + \beta^2 + 1)\beta}{1 + \beta^2} x \right)^n \Psi_0(x) \\ = & (-C_{\alpha, \beta})^n \exp \left(\frac{\alpha^2 - i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x^2 \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{d}{dx} \right)^n \left\{ \exp \left(-\frac{\alpha^2 - i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x \right) \Psi_n(x) \right\} \\
& = A_{\alpha,\beta} (-C_{\alpha,\beta})^n \exp \left(\frac{\alpha^2 - i(\alpha^2 + \beta^2 + 1)\beta}{2(1 + \beta^2)} x \right) \left(\frac{d}{dx} \right)^n \exp \left(-\frac{\alpha^2}{1 + \beta^2} x \right).
\end{aligned}$$

This completes the proof. \square

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